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Exactly solvable $gl(m/n)$ Bose–Fermi systems

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Abstract

A simple SUSY Bose–Fermi Hamiltonian and a class of hard-core Bose–Fermi Hamiltonians with high order terms constructed by using the $gl(m/n)$ generators are shown to be exactly solvable. Excitation energies and corresponding wavefunctions are obtained by using a simple algebraic Bethe ansatz.

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Supersymmetry (SUSY) provides a unified description of bosons and fermions. In particle physics, SUSY offers a possible way to achieve grand unification [1, 2]. SUSY was first studied in the simplest case of SUSY quantum mechanics by Witten, and Cooper and Freedman as a testing ground for the non-perturbative approaches to searching for SUSY breaking in field theory [3–5]. It was realized that SUSY gives insight into the factorization method [6] which was the first method to classify analytically solvable potentials. Over the past few decades, the ideas of SUSY have stimulated new approaches in other branches of physics. For example, evidence has been found for a dynamical SUSY relating even–even and even–odd nuclei [7, 8]. There have also been applications of SUSY in atomic, condensed matter and statistical physics [9–13]. Many researchers extended ideas of SUSY quantum mechanics to other cases and to systems with large numbers of particles with a motivation to understand potential problems of widespread interests [14–16].

In [17], a mean-field plus extended pairing interaction Hamiltonian with many-pair interaction terms was proposed. Though the model contains high order interaction terms, it was shown to be exactly solvable. In this paper, it will be shown that a simple Bose–Fermi SUSY Hamiltonian and an extension of the model including a class of boson–boson, fermion–fermion and boson–fermion interactions are also exactly solvable.

Let $A_i^+ = (A_i)^\dagger$ and A_i be the operator for creating and annihilating a boson or a fermion in the i th level. For simplicity, we assume

$$A_i^+ = \begin{cases} b_i^\dagger & \text{for } i = 1, 2, \dots, m, \\ f_i^\dagger & \text{for } i = m + 1, m + 2, \dots, m + n, \end{cases} \quad (1)$$

where b_i^\dagger , and f_i^\dagger are creation operators for bosons and fermions, respectively, which satisfy the following commutation $[\cdot, \cdot]_-$ or anti-commutation $[\cdot, \cdot]_+$ relations:

$$\begin{aligned} [b_i, b_j^\dagger]_- &= \delta_{ij}, & [b_i, b_j]_- &= 0, & [b_i, f_j^\dagger]_- &= 0, \\ [b_i, f_j]_- &= 0, & [f_i, f_j^\dagger]_+ &= \delta_{ij}, & [f_i, f_j]_+ &= 0. \end{aligned} \quad (2)$$

Using these operators, one can construct generators of the Lie superalgebra $gl(m/n)$ with

$$E_{ij} = A_i^+ A_j \quad (3)$$

for $1 \leq i, j \leq m+n$, which satisfies the graded commutation relations [18]

$$[E_{AB}, E_{CD}] = \delta_{BC} E_{AD} - \delta_{AD} (-1)^{(\sigma_A - \sigma_B)(\sigma_C - \sigma_D)/4} E_{CB}, \quad (4)$$

where $1 \leq A, B, C, D \leq m+n$, and

$$\sigma_i = \begin{cases} +1 & \text{for } 1 \leq i \leq m, \\ -1 & \text{for } m+1 \leq i \leq m+n. \end{cases} \quad (5)$$

Let $\{\epsilon_j\}$ be a set of independent real parameters with $\epsilon_i \neq \epsilon_j$ for $i \neq j$ and $1 \leq i, j \leq n$. One can construct the following Gaudin–Bose or Gaudin–Fermi algebra with

$$N(x) = \sum_{j=1}^n \frac{O_j^+ O_j}{1 - \epsilon_j x}, \quad O(x) = \sum_{j=1}^n \frac{O_j}{1 - \epsilon_j x}, \quad O^+(x) = \sum_{j=1}^n \frac{O_j^+}{1 - \epsilon_j x}, \quad (6)$$

where $O_j = b_j$ or f_j and $O_j^+ = b_j^\dagger$ or f_j^\dagger for Gaudin–Bose or Gaudin–Fermi algebra and x is a complex parameter, which satisfy the following commutation or anti-commutation relations:

$$\begin{aligned} [N(x), O^+(y)]_- &= \frac{1}{x-y} (x O^+(x) - y O^+(y)), \\ [N(x), O(y)]_- &= -\frac{1}{x-y} (x O(x) - y O(y)), \\ [O(x), O(y)]_\pm &= 0, \quad [O(x), O^+(y)]_\pm = \frac{1}{x-y} (x f(x) - y f(y)), \end{aligned} \quad (7)$$

where commutation and anti-commutation relations are applied to Bose and Fermi cases, respectively,

$$f(x) = \sum_{j=1}^n \frac{1}{1 - \epsilon_j x}. \quad (8)$$

Then, using (7), one can prove that the Hamiltonian

$$\hat{H} = \frac{\partial}{\partial x} N(x)|_{x=0} + G O^+(0) O(0), \quad (9)$$

where G is a real parameter, is exactly diagonalized under the Bethe ansatz wavefunction

$$|k; \{x_i\}\rangle = O^+(x_1) O^+(x_2) \cdots O^+(x_k) |0\rangle, \quad (10)$$

where $|0\rangle$ is the vacuum state satisfying $O_i |0\rangle = 0$, with energy eigenvalues given by

$$E_k = \sum_{i=1}^k \frac{1}{x_i}, \quad (11)$$

and the set of parameters $\{x_i\}$ satisfy the Bethe ansatz equations,

$$\frac{1}{x_i} = G f(x_i) \quad \text{for } i = 1, 2, \dots, k. \quad (12)$$

Besides the obvious trivial root when $x_i \rightarrow \infty$, there are exactly n different roots of (12). For the Bose cases, there is no restriction to the roots of (12) in (10) and (11), so x_1, x_2, \dots, x_k can be taken as any roots of (12). For the Fermi cases, due to the Pauli principle $(O^+(x))^2 = 0$ and no pair of roots among x_1, x_2, \dots, x_k can be taken to be the same. This means there are $n!/(n-k)!k!$ solutions in total. Hence, both the Bose and Fermi Hamiltonians given in (9) are exactly solvable [19].

Next, we assume that there are m non-degenerate boson levels ϵ_i ($i = 1, 2, \dots, m$) and n non-degenerate fermion levels with energies ϵ_i ($i = m + 1, m + 2, \dots, m + n$). Using the same procedure, one can prove that a Hamiltonian constructed by using the generators E_{ij} of the Lie superalgebra $gl(m/n)$ with

$$\hat{H} = \sum_{i=1}^{m+n} \epsilon_i E_{ii} + G \sum_{1 \leq i, j \leq m+n} E_{ij} \tag{13}$$

is also solvable when $\epsilon_i \neq \epsilon_j$ for any $i \neq j$. For k -particle excitations, the wavefunction of the system can be written as

$$|k; \{x_i\}\rangle = A^+(x_1)A^+(x_2) \cdots A^+(x_k)|0\rangle, \tag{14}$$

where

$$A^+(x) = \sum_{i=1}^{m+n} \frac{A_i^+}{1 - \epsilon_i x}. \tag{15}$$

Eigen-energies of (13) are still given by (11). The c -numbers x_1, x_2, \dots, x_k in this case should satisfy

$$\frac{1}{x_i} = G \sum_{j=1}^{m+n} \frac{1}{1 - \epsilon_j x} \quad \text{for } i = 1, 2, \dots, k. \tag{16}$$

In this case, similar to the Bose cases shown in (10), x_1, x_2, \dots, x_k can be taken as any roots of (16). Since $f_i^{+2} = 0$, the Fermion sectors will vanish automatically when the Pauli principle is violated.

Furthermore, there exists another exactly solvable hard-core Bose case. Hard-core bosons, like fermions, satisfy the restriction with $(b_i^\dagger)^2 = 0$. Hence, the operators A_i^+ ($i = 1, 2, \dots, m + n$) in this case uniformly satisfy the nilpotent condition

$$(A_i^+)^2 = 0. \tag{17}$$

Thus, the solutions (11), (14), (15) and (16) provide complete solutions of (13) with no pair of the roots among x_1, x_2, \dots, x_k being taken the same.

Finally, similar to [17], one can extend the $gl(m/n)$ Hamiltonian (13) for the hard-core Bose case by introducing high-order terms. The generators of the Gaudin–Bose or Gaudin–Fermi algebra $\{O^+(x_i), O(x_i^*)\}$ ($i = 1, 2, \dots, \tau$), where $\{x_i\}$ fulfil (8) and (12) with replacing n by τ , satisfy the following relations:

$$\begin{aligned} [O(x_\mu^*), O^+(x_\nu)]_{\pm} &= \delta_{\mu\nu} J(x_\mu), \\ J(x_\mu) &= \sum_{j=1}^{\tau} \frac{1}{(1 - \epsilon_j x_\mu^*)(1 - \epsilon_j x_\mu)}, \end{aligned} \tag{18}$$

where $\tau = m$ or n for Bose or Fermi case. Hence, the operators $\{O^+(x_i), O^+(x_i^*)\}$ can be normalized as $\{O^+(x_i)\}$ with

$$O^+(x_i) = \sqrt{\frac{1}{J(x_i)}} O^+(x_i), \tag{19}$$

and

$$[\mathcal{O}(x_\mu), \mathcal{O}^+(x_\nu)]_\pm = \delta_{\mu\nu}. \quad (20)$$

Using the normalized operators, we may construct a set of commutative pairwise operators,

$$B^+(x_i, x_j) = \mathcal{O}^+(x_i)\mathcal{O}^+(x_j) \quad (21)$$

with $i \neq j$. The primitive set of such operators is $\{B^+(x_1, x_2), B^+(x_3, x_4), \dots, B^+(x_{2[\tau/2]-1}, x_{2[\tau/2]})\}$, where $[q]$ stands for the integer part of q . Let S_τ be the permutation group operating among the indices $\{1, 2, \dots, \tau\}$, and ω be a representative in the decomposition $S_\tau \downarrow (S_2 \otimes)^{[\tau/2]-1} S_2$ with

$$S_\tau = \sum_{\omega} \omega((S_2 \otimes)^{[\tau/2]-1} S_2). \quad (22)$$

It is obvious that $\{B^+(x_{\omega(1)}, x_{\omega(2)}), B^+(x_{\omega(3)}, x_{\omega(4)}), \dots, B^+(x_{\omega(2[\tau/2]-1)}, x_{\omega(2[\tau/2])})\}$ gives another set of commutative pairwise operators. So there are $\tau!/(2^{[\tau/2]}[\tau/2]!)$ different such sets in total. Let

$$P_i^+ = \mathcal{O}^+(x_{\omega(2i-1)})\mathcal{O}^+(x_{\omega(2i)}) \quad (23)$$

for $i = 1, 2, \dots, [\tau/2]$, which satisfy the following commutation relations:

$$[P_i^+, P_j^+]_- = 0, \quad [P_i, P_j]_- = \delta_{ij}(1 \pm (n_{2i-1} + n_{2i})), \quad (24)$$

where the $+$ or $-$ sign in (23) corresponds to the Bose or Fermi cases, and $n_i = \mathcal{O}^+(x_{\omega(i)})\mathcal{O}(x_{\omega(i)})$. In the following, for simplicity, we set ω to be the identity operation. The procedure and results for other choices of ω are similar. Due to the hard-core restriction, the relation (24) enables us to show that the extended $gl(m/n)$ Hamiltonian including pairwise high-order interactions with

$$\begin{aligned} \hat{H} = & \sum_{i=1}^{m+n} \epsilon_i E_{ii} + G \sum_{1 \leq i, j \leq m+n} E_{ij} + \kappa \sum_{ij} \mathcal{P}_i^+ \mathcal{P}_j \\ & + \kappa \sum_{\mu=2}^{\infty} \frac{1}{(\mu!)^2} \sum_{i_1 \neq \dots \neq i_{2\mu}} \mathcal{P}_{i_1}^+ \mathcal{P}_{i_2}^+ \dots \mathcal{P}_{i_\mu}^+ \mathcal{P}_{i_{\mu+1}} \dots \mathcal{P}_{i_{2\mu}}, \end{aligned} \quad (25)$$

where κ is a real parameter, is also exactly solvable, where

$$\mathcal{P}_i^+ = \mathcal{A}^+(x_{2i-1})\mathcal{A}^+(x_{2i}) \quad (26)$$

for $i = 1, 2, \dots, [(m+n)/2]$, with

$$\mathcal{A}^+(x_j) = \sqrt{\frac{1}{\mathcal{J}(x_j)}} \mathcal{A}^+(x_j), \quad (27)$$

where

$$\mathcal{J}(x_\mu) = \sum_{j=1}^{m+n} \frac{1}{(1 - \epsilon_j x_\mu^*)(1 - \epsilon_j x_\mu)}. \quad (28)$$

Let $|j_1, j_2, \dots, j_p\rangle$ be the pairwise vacuum state that satisfies

$$\mathcal{P}_i |j_1, j_2, \dots, j_p\rangle = 0 \quad (29)$$

for $1 \leq i \leq [(m+n)/2]$, where j_1, j_2, \dots, j_p indicate that those p levels are occupied by single particles. For example, when $m+n = 4$, the pairwise vacuum states in this case are

$$\{|0\rangle, |1\rangle, |2\rangle, |3\rangle, |4\rangle, |13\rangle, |14\rangle, |24\rangle\}, \quad (30)$$

where $|0\rangle$ is the vacuum state, and $|i\rangle = \mathcal{A}^+(x_i)|0\rangle$, etc.

Using the procedure similar to that used in [17], one can prove that k -pair eigenstate of (25) can be written as

$$|k; \zeta; j_1, j_2, \dots, j_p\rangle = \sum'_{1 \leq i_1 < i_2 < \dots < i_k < [(m+n)/2]} C_{i_1 i_2 \dots i_k}^{(\zeta)} \mathcal{P}_{i_1}^+ \mathcal{P}_{i_2}^+ \dots \mathcal{P}_{i_k}^+ |j_1, j_2, \dots, j_p\rangle, \quad (31)$$

where the prime indicates that the sum only runs over those i_μ that are not related to any one of the $\{j_\nu\}$ in the pairwise vacuum state $|j_1, j_2, \dots, j_p\rangle$ due to the hard-core restriction. In addition, due to the hard-core restriction and the strict ordering of the sum in (31), the term with $\pm(n_{2i-1} + n_{2i})$ given by the commutation relation in (24) has no effect when \mathcal{P}_i is applied to (31). Thus, the expansion coefficient $C_{i_1 i_2 \dots i_k}^{(\zeta)}$ can be expressed as

$$C_{i_1 i_2 \dots i_k}^{(\zeta)} = \frac{1}{1 - y^{(\zeta)} \sum_{\mu} \phi_{i_\mu}}, \quad (32)$$

where $y^{(\zeta)}$ is a c -number to be determined, and $\phi_i = 1/x_{2i-1} + 1/x_{2i}$. The k -pair excitation energies of (25) are given by

$$E_k^{(\zeta)} = \sum_{\mu=1}^p \frac{1}{x_{j_\mu}} + \frac{1}{y^{(\zeta)}} + \kappa(k-1) \quad (33)$$

and the variable $y^{(\zeta)}$ is given by

$$\frac{1}{y^{(\zeta)}} - \kappa \sum'_{1 \leq i_1 < i_2 < \dots < i_k < [(m+n)/2]} C_{i_1 i_2 \dots i_k}^{(\zeta)} = 0. \quad (34)$$

It is clear that there are $(m+n)!/(2^{[(m+n)/2]} [(m+n)/2]!)$ different Hamiltonians similar to that given by (25), which can be obtained by replacing the operators \mathcal{P}_i^+ in (25) with $\mathcal{P}_{\omega(i)}^+$ for $i = 1, 2, \dots, [(m+n)/2]$. These Hamiltonians are all exactly diagonalizable by using the procedure outlined above.

There are several special cases of Hamiltonian (25). When $\kappa = 0$, the Hamiltonian (25) contains only one-body terms, which describes a system with spinless fermions and hard-core bosons hopping among different orbits. When $G = 0$ and $\kappa \neq 0$, the Hamiltonian (25) describes a system of hard-core bosons and spinless fermions hopping among different orbits with high order pairing interactions, which is similar to the pure hard-core boson case shown in [17]. The general Hamiltonian given in (25) can be regarded as a special solvable SUSY case for certain trapped boson–fermion mixtures with high order pairing interactions [20, 21].

In summary, we have shown that a simple $gl(m/n)$ Bose–Fermi Hamiltonian and a class of hard-core $gl(m/n)$ Bose–Fermi Hamiltonians with high order interaction terms are exactly solvable. Excitation energies and corresponding wavefunctions can be obtained by using a simple algebraic Bethe ansatz, which provides new classes of solvable models with dynamical SUSY. The results should be helpful in searching for other exactly solvable SUSY quantum many-body models and understanding the nature of the exactly or quasi-exactly solvability. It is obvious that such Hamiltonians with only Bose or Fermi sectors are also exactly solvable by using the same approach. Indeed, it was shown recently that a similar Fermi version of the model with high-order terms is applicable to well-deformed nuclei, and may also be useful in studying pairing phenomena in metallic clusters of nanoscale size [17]. The Bose and Fermi version of the model may also be useful in studying hard-core Bose– and Fermi–Hubbard models, Bose–Einstein condensates, etc. An extension to including spin degrees of freedom in the model is also possible. Work along these lines is in progress.

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